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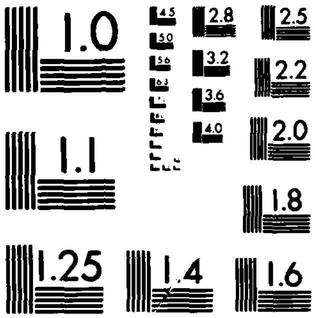

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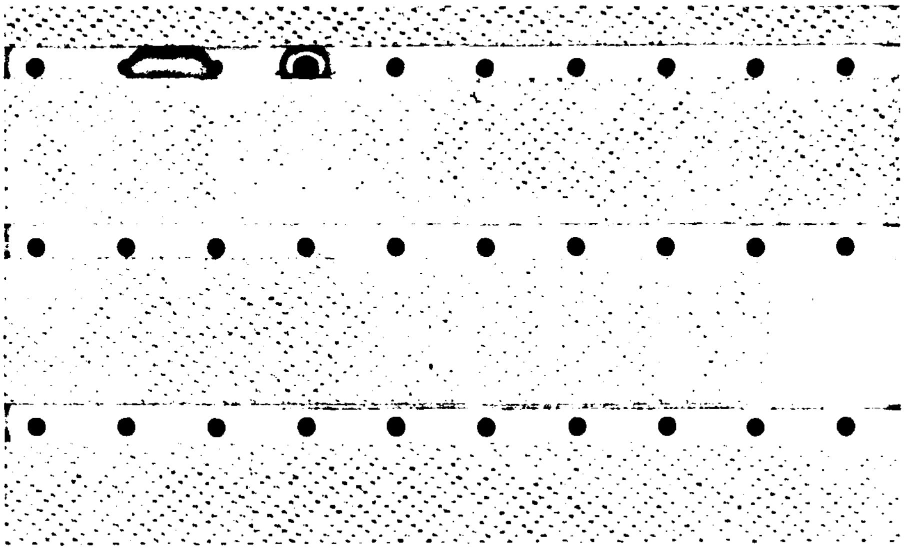

CWP-014

TWO-AND-ONE-HALF DIMENSIONAL IN-PLANE WAVE PROPAGATION

by

Norman Bleistein

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ABSTRACT

The purpose of this paper is to collect certain wave propagation results in two-and-one-half dimensions — defined as three dimensional propagation in a medium that has variations in two dimensions only. The results of interest are for sources and receivers in the plane determined by the two directions of parameter variation. The objective of this work is to reduce the analysis of the in-plane propagation to two dimensional analysis while retaining — at least asymptotically — the proper three dimensional geometrical spreading. We do this for the free space Green's function and for the Kirchhoff approximate upward scattered field from a single reflector. In both cases, we carry out a derivation under the assumption of a background velocity with two dimensional — $c(x, z)$ — variation, we specialize the results to a constant background velocity and a depth dependent background velocity. For the convenience of the user we have included a glossary and two tables of equation numbers to help in finding specific results.

GLOSSARY

$A(\underline{x})$	amplitude of ray theoretic Green's function in 3D, constant density.
$A_2(\underline{x})$	amplitude of ray theoretic Green's function in 2D, constant density.
α	ray parameter labeling rays; polar angle of initial direction in (x,y)-plane.
B	amplitude of ray theoretic Green's function in 3D, variable density.
B_2	amplitude of ray theoretic Green's function in 2D, variable density.
β	ray parameter labeling rays; azimuthal angle with respect to z; becomes polar angle in (x,z) plane when $y = 0$.
$c(\underline{x})$	background velocity.
$c(z)$	background velocity depending on z alone.
$c_+(c_-)$	propagation velocity below (above) reflector, S.
J	Jacobian of mapping via rays in 3D, $\partial(x_1,x_2,x_3)/\partial(\sigma,\alpha,\beta)$.
K	Jacobian of mapping via rays in 2D, $\partial(x_1,x_3)/\partial(\sigma,\beta)$, $\alpha = 0$.
$n(z)$	index of refraction with $c(z)$ background velocity, $n(z) = c(\xi_3)/c(z)$.
$\underline{p}$	$\nabla\tau$, $\underline{p}\cdot\underline{p} = 1/c^2$.
R	ray theoretic reflection coefficient defined by (46), specialized to constant background velocity in (62).
r	$\sqrt{(x_1 - \xi_1)^2 + (x_3 - \xi_3)^2}$
ρ	variable density.
σ	ray parameter for which $\dot{\underline{x}}^2 = 1/c^2$ along a ray.
$\tau(\underline{x})$	traveltime.

$u_S(\underline{\eta};\underline{\xi})$	upward scattered wave from a reflector S due to a point source at $\underline{\xi}$ evaluated at $\underline{\eta}$.
(x, y, z)	Cartesian coordinates.
(x_1, x_2, x_3)	Cartesian coordinates.
$\underline{x}$	Cartesian coordinates.
$\underline{\xi}$	$(\xi_1, 0, \xi_3)$, source point for Green's function.
z_β	turning point; $n(z_\beta) = \sin\beta$.

1. INTRODUCTION

The gathering of seismic data over a line on the surface of the earth or ocean — rather than over a planar or areal array — is still a pre-eminent methodology. Consequently, methods of reflector imaging and earth parameter estimation — migration, structural inversion, seismic inversion — must be based on models of the substructure and wave propagation which can produce meaningful results from this dimensionally constrained data.

Finite difference wave equation migration [Claerbout, [1976] deals with this difficulty by assuming a two dimensional model, both for the substructure and the wave propagation. The two dimensions are the depth — z or x_3 — and the *transverse range* along the line of the seismic experiments x or x_1. While it might be reasonable to assume that the out-of-plane variations are small enough to be neglected, the implementation of a two dimensional wave equations guarantees that the amplitudes generated will be progressively more inaccurate with traveltime due to improper characterization by the two dimensional wave equation of the geometrical spreading effects in three dimensions.

On the other hand, Kirchhoff migration [Schneider, 1978], k-f migration [Stolt, 1978], and Born Inversion [Cohen and Bleistein, 1979], among others, properly allow *for* three dimensional propagation of waves — within the accuracy of the assumptions of these methods. However, neither these methods nor any other can treat three dimensional variation in the substructure when only a linear survey is carried out on the surface. The limitation of the data is accomodated by assuming that the earth parameters

vary only with the in-plane variables — (x, z) -- and are independent of the out-of plane variable, y or x_2.

Thus, the earth variations are essentially two dimensional — cylindrical in three dimensions — while the propagation is three dimensional. A few years ago, I heard a presentation by G. Hohmann on an electromagnetic problem with the same type of geometry. He called this case two-and one-half dimensional -- $2^1/_2$D. I have adopted that terminology, as well.

In $2^1/_2$D one is almost always interested in only in-plane values of the wave field; that is, values of wavefield with y or x_2 equal to zero. Certainly one needs only in-plane values of the earth parameters. Thus, except for the effect of three dimensional spreading, the problems of interest are essentially two dimensional.

In research in our own group, we are constantly in need of in-plane evaluations of $2^1/_2$D asymptotic wave fields both for inversion algorithms and for direct modeling. We find that we can reduce such wave fields to 2D wave fields multiplied by appropriate factors, also determined totally in 2D. The purpose of this note is to derive and tabulate some of those results. The starting point for these results is either three dimensional ray theory or a generalized version of the Kirchhoff approximation for the upward scattered wave for a single reflector. These results are available from many sources. However, I shall use my own book [Bleistein, 1984] -- referred to below as MMWP -- because I can find those formulas most easily there.

We shall state results for a $c(x,z)$ background velocity and specialize those to constant background velocity where the results become explicit and to a $c(z)$ background velocity where the results are somewhat more explicit than in the general case. For most of the paper, we discuss the case of constant density and variable soundspeed. However, in Section 5 we describe the necessary modifications for variable density. For the convenience of the user I have included a glossary of the notation of this paper and two tables of equation numbers to help in finding specific results.

2. ASYMPTOTIC $2^1/_2$D GREEN'S FUNCTION

We consider the following problem for u:

$$\nabla^2 u + \frac{\omega^2}{c^2} u = -\delta(x_1 - \xi_1)\ \delta(x_2)\ \delta(x_3 - \xi_3) \quad . \tag{1}$$

In this equation, ∇^2 is the Laplacian in the variables $\underline{x} = (x, y, z) = (x_1, x_2, x_3)$ and $c = c(x, z) = c(x_1, x_3)$. We use the convention that $z = x_3$ is positive downward. We propose to solve this problem by ray methods [Keller, 1958; Lewis and Keller, 1964]. Our objective, here, will be to reduce this ray theoretic problem in three dimensions to a problem in two dimensions when $x_2 = 0$ while still retaining the correct solution of the three dimensional problem by appropriate adjustment of the equations and scaling of the solution.

We assume that

$$u(\underline{x}; \omega) \sim A(\underline{x})\ \exp\{i\omega\tau(\underline{x})\} \tag{2}$$

with τ a solution of the <u>eikonal</u> <u>equation</u>,

$$(\nabla\tau)^2 = 1/c^2(\underline{x}) \ , \tag{3}$$

and A a solution of the transport equation

$$2\ \nabla A \cdot \nabla\tau + A\nabla^2\tau = 0 \quad . \tag{4}$$

(See MMWP, Sections 8.2 and 8.3, for details of the derivation of these equations and results below related to ray theory.) The functions, τ and A, must also satisfy the conditions

$$\tau = 0, \quad \text{for} \quad \underline{x} = \underline{\xi} = (\xi_1, 0, \xi_3) \tag{5}$$

$$|\underline{x}-\underline{\xi}|\, A \rightarrow 1/4\pi, \text{ as } \underline{x} \rightarrow \underline{\xi} \quad . \tag{6}$$

We shall solve (3), (5) by the method of characteristics. The <u>characteristic equations</u> or <u>ray equations</u> are

$$\begin{aligned} &\dot{\underline{x}} = \underline{p} \,, && \underline{p} = \nabla\tau, \ \underline{p}^2 = 1/c^2(\underline{x}) \\ &\dot{\underline{p}} = \frac{1}{2}\nabla(1/c^2(\underline{x})) \,, && (\cdot) = \frac{d}{d\sigma} \quad . \\ &\dot{\tau} = 1/c^2(\underline{x}) \,, && \end{aligned} \tag{7}$$

The solution we seek is the <u>conoidal solution</u> for which the rays emanate from a point, $\underline{\xi}$. We require initial data at $\sigma = 0$ for each of the seven unknowns in (7). The data for τ is given in (5), since the rays emanate from $\underline{\xi}$, the initial values of $\underline{x}$ are known. For the conoidal solution, the

initial values of $\underline{p}$ are not known, except that they must be constrained by the eikonal equation, (3), itself: $p^2 = 1/c^2$. Thus, there are two other parameters which serve to label a ray by its initial direction. Therefore the initial data or <u>ray data</u> for (7) is

$$\underline{x}(0) = \underline{\xi}$$

$$\underline{p}(0) = 1/c(\underline{\xi})\ (\cos\alpha\sin\beta,\ \sin\alpha\sin\beta,\ \cos\beta)\quad . \tag{8}$$

$$\tau(0) = 0$$

The solution of ordinary differential equations (7) and the ray data (8) is a family of rays with parameter σ, distinguished from one another by the choice of the parameters α and β. Along each ray, the value of τ is known, as well.

The specific assumption that c is independent of x_2 allows us to obtain a part of this solution in closed form. From (7), the equation for p_2 is

$$\dot{p}_2 = \frac{\partial}{\partial x_2}\ \frac{1}{c^2} = 0\quad . \tag{9}$$

Thus, p_2 is constant on each ray, that is, independent of σ. From the ray data in (8), we know the value of that constant,

$$p_2 = \frac{1}{c(\underline{\xi})} \sin\alpha\ \sin\beta\quad . \tag{10}$$

With p_2 independent of σ, the equation for x_2 in (7) now becomes

particularly simple. Taking account of the initial data in (8), we find that

$$x_2 = \frac{\sigma}{c(\underline{\xi})} \sin\alpha \ \sin\beta \ . \qquad (11)$$

We remark that $x_2 = 0$ for any choice of (x_1, x_3) only if we choose those rays for which $\sin\alpha = 0$.

The transport equation, (4), can also be written as an ordinary differential equation in the ray parameter, σ. The solution of that equation is [MMWP (8.3.12)]:

$$A = \frac{\sqrt{\sin\beta}}{4\pi \sqrt{c(\underline{\xi}) J(\sigma, \alpha, \beta)}} \ . \qquad (12)$$

In this equation, J is the Jacobian of the transformation from $\underline{x}$ to (σ, α, β) via the solution of the ray equations, (7) and (8):

$$J = \left| \frac{\partial(x_1, x_2, x_3)}{\partial(\sigma, \alpha, \beta)} \right| \ . \qquad (13)$$

The following changes in notation from the refernce have been made here: (i) $\gamma_1 \rightarrow \beta$, (ii) $p(\underline{x}_0) \rightarrow 1/c(\underline{\xi})$; (iii) $p(\underline{x}(\sigma))\ J(\sigma) \rightarrow J(\sigma, \alpha, \beta)$. For the last of these, we have used (8.2.29) with $\lambda = 1$ to redefine J as in (13).

The solution (2), with τ a solution of the system (7) and (8) and A given by (12), is valid up to the first zero of J for $\sigma \neq 0$.

3. Asymptotic In-Plane $2^1/_2$D Green's Function

We shall now consider the results of the previous section for the case $x_2 = 0$. We shall refer to this case as the <u>in-plane</u> solution. As noted above, setting $x_2 = 0$ is equivalent to setting $\alpha = 0$ in (8) and (11).

Let us consider the remaining equations in (7) and under the assumption that $\alpha = 0$:

$$
\begin{aligned}
\dot{x}_1 &= p_1 , & x_1(0) &= \xi_1 ; \\
\dot{x}_3 &= p_3 , & x_3(0) &= \xi_3 ; \\
\dot{p}_1 &= \frac{1}{2}\frac{\partial}{\partial x_1}\left[\frac{1}{c^2(x_1, x_3)}\right], & p_1(0) &= \frac{\sin\beta}{c(\underline{\xi})} ; \\
\dot{p}_3 &= \frac{1}{2}\frac{\partial}{\partial x_3}\left[\frac{1}{c^2(x_1, x_3)}\right], & p_3(0) &= \frac{\cos\beta}{c(\underline{\xi})} ; \\
\dot{\tau} &= \frac{1}{c^2(x_1, x_3)} , & \tau(0) &= 0 .
\end{aligned}
\tag{14}
$$

This is a closed system of equations for which $p_1^2 + p_3^2 = 1/c^2(\underline{x})$, since $p_2 = 0$. In fact, it is just the system of ray equations for solving the eikonal equation in two dimensions. Hence, an algorithm for solving ray equations in 2D may be applied to determine the in-plane rays and phase. I do not believe that this would surprise anyone who has read this far.

Let us now consider the solution (12) for the amplitude where $\alpha = 0$. In particular, from (11), we can calculate the elements of the second row of the matrix arising in J, (13):

$$\frac{\partial x_2}{\partial \sigma} = \frac{1}{c(\xi)} \sin\alpha \sin\beta = 0, \quad \alpha = 0\ ;$$

$$\frac{\partial x_2}{\partial \alpha} = \frac{\sigma}{c(\xi)} \cos\alpha \sin\beta = \frac{\sigma}{c(\xi)} \sin\beta, \quad \alpha = 0\ ; \tag{15}$$

$$\frac{\partial x_2}{\partial \beta} = \frac{\sigma}{c(\xi)} \sin\alpha \cos\beta = 0\ , \quad \alpha = 0\ .$$

We see that only the second element is nonzero. Thus, the 3x3 determinant in (13) can be reduced to a 2x2 determinant,

$$J = \frac{\sigma}{c(\xi)} \sin\beta\ K\ ; \tag{16}$$

$$K = \left| \left| \frac{\partial(x_1, x_3)}{\partial(\sigma, \beta)} \right| \right|_{\alpha = 0} \tag{17}$$

The Jacobian, K, can be computed directly from the system (14) for the solution of eikonal equation in two dimensions. We use (16) in (12) and obtain

$$A = \frac{1}{4\pi \sqrt{\sigma K}} \quad . \tag{18}$$

By using (14) and (18), then, we obtain the in-plane $2^1/_2$D asymptotic Green's function totally in terms of solutions of the 2D ray equations. We remark that it was the simple form of this result which motivated the particular choice of ray parameter σ, for which $\dot{\underline{x}}^2 = 1/c^2(\underline{x})$, rather than such alternative ray parameters as arc length on a ray — for which $\dot{\underline{x}}^2 = 1$ — or traveltime — for which $\dot{\underline{x}}^2 = c^2(\underline{x})$. We note, however, that σ is an unphysical variable with dimensions (length)^2/time.

The in-phase representation of the solution (2) now takes the form,

$$u \sim \frac{\exp\{i\omega\tau\}}{4\pi \sqrt{\sigma K}} \quad , \tag{19}$$

with τ, σ and K totally determinable by in-plane calculations. Nonetheless, this solution has the proper geometrical spreading for three dimensional propagation.

The solution (18) can be more precisely related to the solution of the two dimensional transport equation. (The reader should note, however, that the latter amplitude is the response to a line source in three dimensions.) Let us denote that solution in 2D by A_2. The result is given by MMWP (8.3.30), but in different notation. The correct result is

$$A_2 = \frac{\exp\{i\pi/4 \operatorname{sgn} \omega\}}{2\sqrt{2\pi K |\omega|}} \quad . \tag{20}$$

In this equation, we have also included the proper power of ω as given from the discussion above (8.3.30) and we have accounted for negative ω, through the absolute values and the factor $\operatorname{sgn}\omega$.

By comparing (18) and (20), we find that

$$A = A_2 \sqrt{\frac{|\omega|}{2\pi\sigma}} \exp\{-i\pi/4 \operatorname{sgn} \omega\} \quad . \tag{21}$$

Thus, if one has available a 2D ray equation routine which calculates A_2 directly, one can use this routine to generate the in-plane $2^1/_2$D amplitude by employing the complex scaling indicated by (21).

I further take the position that this multiplier is an adequate "quick and dirty" correction to apply to an exact 2D equation solver. The reason is that in geophysical applications, we are almost always interested in the solution many — at least three — wavelengths from the source in a slowly varying background medium. In that limit (high frequency), an asymptotic adjustment will suffice.

It should also be noted that all refraction and reflection effects of the in-plane $2^1/_2$D propagation will remain in-plane. Thus, the same multiplier can be used for a discontinuous $c(x,z)$. However, one must take care to measure σ from the source point and not from the discontinuity

surface. Furthermore, one must then use an amplitude A, which properly accounts for in-plane transmission effects.

For two special choices of $c(x,z)$ we can be more explicit about the relationship between σ and the Cartesian coordinates. In the simplest case, c = constant = c_0, the ray equations, (14), with the indicated ray data can be solved explicitly. First we observe that $\dot{p}_1 = \dot{p}_3 = 0$ and (p_1, p_3) is a constant vector given by the initial data with $c(\xi) = c_0$. We then solve for x_1 and x_3 in (14):

$$x_1 = \xi_1 + c_0^{-1}\sigma \sin\beta, \; x_3 = \xi_3 + c^{-1}\sigma \cos\beta, \; \tau = \sigma/c \tag{22}$$

We now solve for σ in the first two equations

$$\sigma = c_0 r, \; r = \sqrt{(x_1-\xi_1)^2+(x_3-\xi_3)^2} \quad , \tag{23}$$

and find, upon substitution into the third equation,

$$\tau = r/c_0 \tag{24}$$

Furthermore, we can find K by applying the definition (17) to the equations for the rays in (22). The result is

$$K = \sigma/c_0^2 = r/c_0 \tag{25}$$

We use (23) and (25) in (18) to obtain

$$A = (4\pi r)^{-1} \quad . \tag{26}$$

By inserting (24) and (26) in (19) we obtain the exact three dimensional Green's function, which indeed, is equal to its leading order asymptotic expansion. This serves as a simple check on the method.

As a second example, let us suppose that c is a function of z above, $c = c(z) = c(x_3)$. In this case, it is easier to recast (14) as a system of equations in z and express σ as a function of z, as well. We rewrite (14) as

$$\begin{aligned}
&\frac{dx_1}{dz} = \frac{p_1}{p_3} \quad , \qquad x_1(\xi_3) = \xi_1 \; , \\
&\frac{dp_1}{dz} = 0 \quad , \qquad p_1(\xi_1) = \frac{\sin\beta}{c(\xi_3)} \; , \\
&\frac{dp_3}{dz} = \frac{1}{2p_3}\frac{d}{dz}\left[\frac{1}{c^2(z)}\right] , \qquad p_3(\xi_3) = \frac{\cos\beta}{c(\xi_3)} \; , \\
&\frac{d\tau}{dz} = \frac{1}{c^2(z)} \quad , \qquad \tau(\xi_3) = 0 \; , \\
&\frac{d\sigma}{dz} = \frac{1}{p_3} \quad , \qquad \sigma(\xi_3) = 0 \quad .
\end{aligned} \tag{27}$$

From the second equation, here we see that p_1 is a constant on each ray, given by its initial value:

$$p_1 = \frac{1}{c(\xi_3)} \sin\beta \quad . \tag{28}$$

This is Snell's law for a medium with depth-dependent velocity. Since (p_1, p_3) satisfy the eikonal equation in the form in (7), we conclude that

$$p_3 = \sqrt{\frac{1}{c^2(z)} - \frac{1}{c^2(\xi_3)} \sin^2\beta} \quad . \tag{29}$$

We have chosen the positive square root, here, because we are interested in rays which are directed downward; that is, z must increase with σ. At $z = \xi_3$, the square root is real for all β. For $|\beta| < \pi/2$, p_3 will remain real in some neighborhood of ξ_3. We continue our analysis in that region.

We can now proceed to solve the remaining equations in (27)

$$x_1 - \xi_1 = \frac{\sin\beta}{c(\xi_3)} \int_{\xi_3}^{z} \frac{dz'}{\sqrt{\frac{1}{c^2(z')} - \frac{\sin^2\beta}{c^2(\xi_3)}}} ,$$

$$\tau = \int_{\xi_3}^{z} \frac{dz'}{c^2(z') \sqrt{\frac{1}{c^2(z')} - \frac{\sin^2\beta}{c^2(\xi_3)}}} , \tag{30}$$

$$\sigma = \int_{\xi_3}^{z} \frac{dz'}{\sqrt{\frac{1}{c^2(z')} - \frac{\sin^2\beta}{c^2(\xi_3)}}} .$$

It is this last factor which we use in (18) or (17) or (21) to adjust the in-plane Green's function for geometrical spreading out-of-plane.

For computational purposes, it is desirable to express these results in terms of an index of refraction:

$$n(z) = \frac{c(\xi_3)}{c(z)} . \tag{31}$$

Those equations are

$$x_1 - \xi_1 = \sin\beta \int_{\xi_3}^{z} \frac{dz'}{\sqrt{n^2(z') - \sin^2\beta}} ,$$

$$\tau = \frac{1}{c(\xi_3)} \int_{\xi_3}^{z} \frac{n^2(z')dz'}{\sqrt{n^2(z') - \sin^2\beta}} , \tag{32}$$

$$\sigma = c(\xi_3) \int_{\xi_3}^{z} \frac{dz'}{\sqrt{n^2(z') - \sin^2\beta}} .$$

To complete the computations for the Green's function for this case, we must determine K, defined by (17). To simplify the computation, we note that

$$\left|\frac{\partial(x_1, x_3)}{\partial(\sigma, \beta)}\right| = \left|\frac{\partial(x_1, z)}{\partial(\beta, z)} \; \frac{\partial(z, \beta)}{\partial(\sigma, \beta)}\right| = \left|\frac{\partial x_1}{\partial\beta} \frac{\partial z}{\partial\sigma}\right| , \tag{33}$$

by application of the chain rule for Jacobi determinants. The derivatives in the last expression are partial derivatives because x_1, z are determined in the plane as functions of the two variable β, z. We calculate $\partial x_1/\partial\beta$ from the first equation in (31), $\partial z/\partial\sigma$ is determined from the last equation in (28). The result is that

$$K = \left|\frac{\partial x_1}{\partial\beta} \cdot \frac{\partial z}{\partial\sigma}\right| = \sqrt{n^2(z) - \sin^2\beta}\ \frac{\cos\beta}{c(\xi_3)} \int_{\xi_3}^{z} \frac{n^2(z')dz'}{(n^2(z') - \sin^2\beta)^{3/2}} . \tag{34}$$

We can now use (18), (30) and (32) in (19) to write down the asymptotic $2^1/_2$D solution evaluated in-plane. The result is given parametrically with parameter β. For each coordinate pair, (x_1, z), we determine β from the first equation in (32). We substitute this solution into the second and third equations of (32) to determine τ and σ, and we substitute into (34) to determine K. When all of these results are substituted into (19), u is determined.

We remark that the function K in (34) is never equal to zero for $z \neq \xi_3$. That is, the downward propagating part of the Green's function in a c(z) medium has no caustics. The only pathology that could occur is that for some β, the ray propagates to a depth, z_β, such that $n(z_\beta) = \sin\beta$. If we assume that $n'(z_\beta) \neq 0$, then K has a finite nonzero limit at z_β even though the integral in (34) diverges for $z = z_\beta$. To see why this is so, multiply and divide by $n'(z')$ under the integral sign and integrate by parts. The result is

$$K = \sqrt{n^2(z) - \sin\beta}\ \frac{\cos\beta}{c(\xi_s)} \left[- \frac{n(z)}{n'(z)\sqrt{n^2(z)-\sin^2\beta}} + \frac{n(\xi_s)}{n'(\xi_s)\sqrt{\eta^2(\xi_s)-\sin^2\beta}} \right. \tag{35}$$

$$\left. + \int_{\xi_s}^{z} \left[\frac{n(z')}{n'(z')}\right]' \frac{dz'}{\sqrt{n^2(z') - \sin^2\beta)}} \right]$$

The integral, here, converges even when $z = z_\beta$. Thus, if we now take the limit as $z \rightarrow z_\beta$, we find that

$$\lim_{z \rightarrow z_\beta} K = - \frac{\cos\beta}{c(\xi_s)} \frac{n(z_\beta)}{n_\beta'(z\ )} . \tag{36}$$

To be completely rigorous in this derivation, we should only integrate by parts over an interval from z_0 to z, where z_0 is sufficiently close to z_β to guarantee that $n'(z) \neq 0$ for $z_0 \leq z \leq z_\beta$. Of course, the same result obtains because the remaining integral on the interval form ξ_s to z_0 remains finite as $z \rightarrow z_\beta$ while the square root multiplier approaches zero in that limit.

The point z_β is a <u>turning point</u> for the ray labeled by β. On the continuation of the ray, z decreases. To obtain the extension to this ray continuation, we need only replace the integral over the interval (ξ_s, z) by

a pair of integrals over the intervals (z, z_β) and (ξ_3, z_β):

$$x_1 - \xi_1 = \sin\beta \int_{z_\beta}^{\xi_3} + \int_{z_\beta}^{z} \frac{dz'}{\sqrt{n^2(z') - \sin^2\beta}} ,$$

$$\tau = \frac{1}{c(\xi_3)} \int_{z_\beta}^{\xi_3} + \int_{z_\beta}^{z} \frac{n^2(z')dz'}{\sqrt{n^2(z') - \sin^2\beta}} , \tag{37}$$

$$\sigma = c(\xi_3) \int_{z_\beta}^{\xi_3} + \int_{z_\beta}^{z} \frac{dz'}{\sqrt{n^2(z') - \sin^2\beta}} .$$

The determination of K is not so straightforward. We cannot simply differentiate the first line, here, to determine $\partial x_1/\partial\beta$, because the resulting integral diverges. We must first integrate by parts as was done above to analyze K near z_β.

$$\begin{aligned} x_1 - \xi_1 = \sin\beta \Bigg[& \frac{\sqrt{n^2(z) - \sin^2\beta}}{n(z)n'(z)} + \frac{\sqrt{n^2(\xi_3) - \sin^2\beta}}{n(\xi_3)\; n'(\xi_3)} \\ & - \left[\int_{z_\beta}^{\xi_3} + \int_{z_\beta}^{z} \right] \left[\frac{1}{n(z')n'(z')}\right]' \sqrt{n^2(z') - \sin^2\beta} \;\; dz' \Bigg] , \end{aligned} \tag{38}$$

and then differentiate with respect to β. This leads to the following result for K:

$$K = \sqrt{n^2(z) - \sin^2\beta}\quad \frac{\cos\beta}{c(\xi_s)}$$

$$\cdot \left[\frac{n^2(z) - 2\sin^2\beta}{n(z)n'(z)\sqrt{n^2(z) - \sin^2\beta}} + \frac{n^2(\xi_s) - 2\sin^2\beta}{n(\xi_s)n'(\xi_s)\sqrt{n^2(\xi_s) - \sin^2\beta}} \right.$$

$$\left. - \left[\int_{z_\beta}^{\xi_s} + \int_{z_\beta}^{z} \right] \left[\frac{1}{n(z')n'(z')} \right]' \frac{n^2(z') - 2\sin^2\beta}{\sqrt{n^2(z') - \sin^2\beta}}\, dz' \right] . \tag{39}$$

Again, we must take care not to use this result if $n'(z) = 0$ over some part of the domain. We would revise this result by using the integration by parts technique only over a portion of the domain of integration near z_β in (37) and simply differentiating with respect to β in the remaining integral.

We remark that now K can be zero for an appropriate $c(z)$. That is, beyond the turning point, the ray family can have a caustic.

This completes our discussion of the in-plane analysis of the $2^1/_2$D asymptotic Green's function.

4. The In-Plane Kirchhoff Approximation in $2^1/_2$D.

We shall now consider the following problem in a $c(x,z)$ medium. A point source is located at $(\xi_1, 0, \xi_3)$, a receiver is located at $(\eta_1, 0, \eta_3)$ and a reflecting surface, S, is located at depth. Consistent with our interest in $2^1/_2$D, we assume that the reflector is cylindrical, that is, the reflector is generated by a curve in the (x,z)-plane and straight lines through that curve parallel to the y-axis. Thus, we characterize the reflector by the generating curve C in the (x,z)-plane.

$$C: \; x = x_1(s), \qquad z = x_3(s) \quad . \tag{40}$$

We shall take s to be an arclength variable along C. Our objective is to obtain an asymptotic representation of the upward propagating wave arising in response to the reflector under the assumption that the downward propagating wave has no caustics before impact with the reflector. Our representation is to be given totally in terms of in-plane functions.

The reflector is to be a surface of discontinuity of the velocity profile, with no discontinuities above it. Thus, we will denote by $c_-(s)$ and $c_+(s)$ the limits of $c(x,z)$ from above and below S, respectively.

We begin our analysis with the Kirchhoff integral representation of the upward scattered field [MMWP(8.4.5)]:

$$u_S(\underline{\eta};\underline{\xi}) = \int_S \left[u_S(\underline{x};\underline{\xi}) \frac{\partial G(\underline{x};\underline{\eta})}{\partial n} - G(\underline{x};\underline{\eta}) \frac{\partial u_S(\underline{x};\underline{\xi})}{\partial n} \right] dS \quad . \tag{41}$$

In this equation, $u_S(\underline{\eta};\underline{\xi})$ denotes the response at $\underline{\eta}$ due to a source at $\underline{\xi}$; $G(\underline{x};\underline{\eta})$ is the free space Green's function at $\underline{x}$ due to a source at $\underline{\eta}$; $\partial/\partial n$ is the upward normal derivative on S.

We shall make a number of asymptotic approximations in the integral (41). First, we shall use for G the asymptotic Green's function of Section 2. We cannot, as yet, use the results of Section 3, since the integration in (41) is over a domain which extends out of the plane, $y = 0$. To emphasize the dependence on the source point, we write the solution (2) as

$$G(\underline{x};\underline{\eta}) = A(\underline{x};\underline{\eta}) \exp \{i\omega\tau(\underline{x};\underline{\eta})\} \quad . \tag{42}$$

For the normal derivative of G, we only calculate the leading order term at high frequency, arising from differentiating only the exponent:

$$\frac{\partial G(\underline{x};\underline{\eta})}{\partial n} \sim i\,\omega\,\hat{n}\cdot\nabla\tau(\underline{x};\underline{\eta})\; A(\underline{x};\underline{\eta})\, \exp\{i\omega\tau(\underline{x};\underline{\eta})\} \tag{43}$$

Here, $\hat{n}$ is an upward unit normal vector and the gradient is to be calculated with respect to the variables, $\underline{x}$. It would be inconsistent to retain the term arising from the differentiation of A, itself, since this is of the

same (lower) order as a term arising from a first order correction to A multiplied by $i\omega\hat{n}\cdot\nabla\tau$.

For $u_S(\underline{x};\underline{\xi})$ on S, we will use a ray theoretic approximation. This field is the reflected wave on S due to the incidence of the wave in response to the point source at $\underline{\xi}$, say, $u_I(\underline{x};\underline{\xi})$. This function is also a Green's function:

$$u_I(\underline{x};\underline{\xi}) = G(\underline{x};\underline{\xi}) \quad . \tag{44}$$

The value of $u_S(\underline{x};\underline{\xi})$ on S is just the <u>ray data</u> for the upward reflected wave. This ray data is derived in [MMWP, Section 8.3]. We find that

$$u_S(\underline{x};\underline{\xi}) = Ru_I(\underline{x};\underline{\xi}) \ , \quad \underline{x} \text{ on S} \quad , \tag{45}$$

where the ray theoretic reflection coefficient is given by [MMWP(8.3.47)]:

$$R = \frac{\partial\tau_I/\partial n - \left[\operatorname{sgn} \partial\tau_I/\partial n\right] \sqrt{c_+^{-2} - c_-^{-2} + [\partial\tau_I/\partial n]^2}}{\partial\tau_I/\partial n + \left[\operatorname{sgn} \partial\tau_I/\partial n\right] \sqrt{c_+^{-2} - c_-^{-2} + [\partial\tau_I/\partial n]^2}} \tag{46}$$

In this equation, we have used the notation

$$\tau_I = \tau(\underline{x};\underline{\xi}) \ , \ \frac{\partial\tau_I}{\partial n} = \hat{n}\cdot\nabla\tau(\underline{x};\underline{\xi}) \quad . \tag{47}$$

In summary, then, we use on S

$$u_S(\underline{x};\underline{\xi}) \sim R\, A(\underline{x};\underline{\xi}) \exp\{i\omega\tau(\underline{x};\underline{\xi})\} \quad ,$$

$$\frac{\partial u_S(\underline{x};\underline{\xi})}{\partial n} \sim -\, i\omega\,\hat{n}\cdot\nabla\tau(\underline{x};\underline{\xi}) R\, A(\underline{x};\underline{\xi}) \exp\{i\omega\tau(\underline{x};\underline{\xi})\} \quad . \tag{48}$$

The minus sign in the second line here is not obvious. It arises from the fact that the normal component of the gradient of the phase on the reflected wave must be opposite in sign to the normal component on the incident wave in order that the incident and reflected waves be oppositely directed with respect to S.

These two approximations constitute a generalization of the approximations credited to Rayleigh, Sommerfeld, Fresnel and Huygens, as well as Kirchhoff. For discussions of the distinctions between these approximations, see Goodman [1968], Kuhn and Alhilali [1977] and Wolf and Marchand [1964]. For simplicity, we shall refer to (46) as the Kirchhoff approximation.

We substitute (40), (41) and (46) into (31) and obtain

$$u_S(\underline{\xi};\underline{\eta}) \sim i\omega \int_S R\, A(\underline{x};\underline{\xi})\, A(\underline{x};\underline{\eta}) \exp\{i\omega[\tau(\underline{x};\underline{\xi})+\tau(\underline{x};\underline{\eta})]\} \cdot \hat{n}\cdot[\nabla\tau(\underline{x};\eta) + \nabla\tau(\underline{x};\underline{\xi})]\, dS \quad . \tag{49}$$

This asymptotic representation is given in terms of the $2^1/_2$D ray theoretic amplitude and phase introduced in Section 2. However, it still requires out of plane computations of amplitude and phase.

We will now reduce the representation (49) to one in which the integrand is given totally in terms of the in-phase amplitude and phase. To do this, we note first that

$$dS = dsdy = ds\ dx_2 \quad . \tag{50}$$

That is, the differential surface area is the product of the differential arc length ds along the generating curve, (40), and the differential out-of-phase coordinate.

We will calculate the integral in $y = x_2$ by the method of stationary phase [MMWP, Chap. 2]. The phase is

$$\Phi(\underline{x};\underline{\xi};\underline{\eta}) = \tau(\underline{x};\underline{\xi}) + \tau(\underline{x};\underline{\eta}) \quad , \tag{51}$$

with first derivative

$$\frac{\partial \Phi}{\partial x_2} = \frac{\partial \tau(\underline{x};\underline{\xi})}{\partial x_2} + \frac{\partial \tau(\underline{x};\underline{\eta})}{\partial x_2} = p_2(\underline{x};\underline{\xi}) + p_2(\underline{x};\underline{\eta}) \quad . \tag{52}$$

In the last equation, we have used the notation of (7), but with arguments that properly relate each p_2 to the corresponding τ.

At the stationary point, $\partial\Phi/\partial x_2 = 0$. Thus, the sum of the p_2's must be zero. From (10) and (11) we see that both p_2's will have the same sign as x_2, for $x_2 \neq 0$. Thus, the sum can only be zero when $x_2 = 0$. That is, the phase has a stationary point in x_2 only for $x_2 = 0$. In terms of the parameters α and β, this occurs at $\alpha = 0$.

We must now determine

$$\frac{\partial^2 \Phi}{\partial x_2^2} = \frac{\partial p_2(\underline{x},\underline{\xi})}{\partial x_2} + \frac{\partial p_2(\underline{x},\underline{\eta})}{\partial x_2} \tag{53}$$

at $x_2 = 0$ or $\alpha = 0$. To do this, we rewrite (11) as

$$x_2 = p_2\ \sigma \tag{54}$$

and differentiate implicitly with respect to x_2:

$$1 = \frac{\partial p_2}{\partial x_2}\ \sigma + p_2\ \frac{\partial \sigma}{\partial x_2} \tag{55}$$

When we evaluate at $p_2 = 0$, we find that

$$\frac{\partial p_2}{\partial x_2} = \frac{1}{\sigma}\ ,\quad p_2 = 0\ . \tag{56}$$

This result applies equally to $p_2(\underline{x};\underline{\xi})$ and $p_2(\underline{x};\underline{\eta})$, except that σ must be properly interpreted as the parameter on the corresponding ray, that is, the parameter connecting $\underline{x}$ to $\underline{\xi}$ or $\underline{x}$ to $\underline{\eta}$. To distinguish between these two ray parameters, we shall denote the first by σ_ξ and the second by σ_η. Thus,

$$\left.\frac{\partial^2 \Phi}{\partial x_2^2}\right|_{x_2=0} = \frac{1}{\sigma_\xi} + \frac{1}{\sigma_\eta}\ . \tag{57}$$

We note that sgn $\partial\Phi/\partial x_2 = +1$ at $x_2 = 0$. This result is needed for the stationary phase evaluation. Furthermore, the entire integrand is to be evaluated at $x_2 = 0$, that is, in-plane. Thus, we may use the results of Section 3, to determine A, τ and σ, now for each of the fixed points, $\underline{\xi}$, and $\underline{\eta}$. The result is

$$u_S\ (\underline{\xi};\underline{\eta}) \sim \frac{\sqrt{2\pi|\omega|}\ \exp\{3\pi i/4\ \mathrm{sgn}\omega\}}{(4\pi)^2} \int_{C,\,x_2=0} \frac{R\ \exp\ \{i\omega[\tau(\underline{x};\underline{\xi})+\tau(\underline{x};\underline{\eta})]\}}{\sqrt{\sigma_\xi K_\xi \sigma_\eta K_\eta(\sigma_\xi^{-1} + \sigma_\eta^{-1})}} \tag{58}$$

$$\cdot\ \hat{n}\cdot[\nabla\tau(\underline{x};\underline{\xi}) + \nabla\tau(\underline{x};\underline{\eta})]\ ds.$$

In this equation, K_ξ and K_η are the in-plane Jacobians defined by (17) for the rays emanating from $\underline{\xi}$ and $\underline{\eta}$, respectively. Evaluation of the integrand at $x_3 = 0$ reduces all of its elements to the in-plane results of Section 3.

It is also worthwhile to specialize (58) to the case of backscatter or zero-offset, $u_S(\underline{\xi};\underline{\xi})$. We can then dispense with the subscripts ξ and η. The result is

$$u_S(\underline{\xi};\underline{\xi}) \sim \frac{\sqrt{|\omega|}\exp\{3\pi i/4 \operatorname{sgn}\omega\}}{8\pi^{3/2}} \int_{C,\,x_3=0} \frac{R\exp\{2i\omega\tau(\underline{x};\underline{\xi})\}}{K\sqrt{\sigma}}\,\hat{n}\cdot\nabla\tau(\underline{x};\underline{\xi})\;ds\;. \tag{59}$$

The results, (58) and (59) are asymptotic representations of the $2^1/_2$D in-plane reflection response for non-zero or zero offset observations, respectively. These formulas are starting points for either numerical computation of the remaining integral or further asymptotic analysis — say, stationary phase in the arc length s. Thus, we have achieved a representation of the in-plane fields in the $2^1/_2$D case with the properties carried out in-plane ($x_3 = 0$) and, nonetheless, the effect of geometrical spreading is accounted for, asymptotically.

As in Section 3, we now consider the special cases of constant background velocity and depth dependent background velocity. For the former case, we use the results (23, (24) and (25) to obtain in place of (58)

$$u_S(\xi;\eta) \sim \frac{\sqrt{2\pi|\omega|}\,\exp\{-\pi i/4\ \mathrm{sgn}\ \omega\}}{(4\pi)^2\ \sqrt{c_0}} \int_{C,\,x_3=0} \frac{R\exp\{i\omega[r_\xi + r_\eta]/c_0\}}{r_\xi r_\eta\ \sqrt{r_\xi^{-1} + r_\eta^{-1}}} \tag{60}$$

$$\cdot\ \hat{n}\cdot[r_\xi + r_\eta]\ ds\quad .$$

In this equation, r_ξ is the distance defined by (22) while $r\eta$ is the same function with ξ replaced by η. Furthermore, $\hat{n}$ is the unit upward normal to C,

$$\hat{r}_\xi = (\xi_1 - x_1, \xi_3 - x_3)/r, \quad \hat{r}_\eta = (\eta_1 - x_1,\ \eta_2 - x_2)/r = -\ c_0\nabla\tau(\underline{x};\eta) \tag{61}$$

and, from (46),

$$R = \frac{\hat{n}\cdot\hat{r}_\xi/c_0 - \sqrt{c_+^{-2} - c_0^{-2} + [\hat{n}\cdot\hat{r}_\xi/c_0]^2}}{\hat{n}\cdot\hat{r}_\xi/c_0 + \sqrt{c_+^{-2} - c_0^{-2} + [\hat{n}\cdot\hat{r}_\xi/c_0]^2}} \quad ; \quad c_0 = c_-\quad . \tag{62}$$

We note, here, that $\hat{n}$ and $\hat{r}_\xi$ are colinear and sgn $\hat{n}\cdot\hat{r}_\xi = +1$. It is apparent from this result that we have momentarily lost the symmetry of u_S in ξ and η. However, were we to apply the method of stationary phase to (60), we would find that the dominant contributions to the field at ξ arise from the specular points, at which $\hat{n}\cdot\hat{r}_\eta = \hat{n}\cdot\hat{r}_\xi$ and the symmetry would be restored. Alternatively, we could redefine (62) as a reflection coefficient R_ξ to

indicate its dependence on the source point ξ and then replace R_ξ by $(R_\xi + R_\eta)/2$. We leave the reflection coefficient as R and leave the option to the reader.

For the case of backscatter, $\xi = \eta$, the result, (60) becomes

$$u_S(\xi,\xi) \sim \frac{\sqrt{|\omega|}\, \exp\{-\pi i/4 \operatorname{sgn} \omega\}}{8\pi^{3/2}\sqrt{c_0}} \int_{C,\, x_3 = 0} \frac{R \exp\{2i\omega\, r/c_0\}}{r^{3/2}}\, \hat{n}\cdot\hat{r}\, ds \quad . \tag{63}$$

with r given by (23).

We now turn to the case of a depth dependent background velocity. In this case, we can do little more than substitute into (58) or (60) the parametric representations given by (32) and (34) — one for the ray family emanating from ξ with ray parameter β_ξ another for the ray family emanating from η with ray parameter β_η.

This completes the discussion of the asymptotic $2^1/_2$D in-plane upward scattered field.

5. The Affect of Variable Density

It is fairly straightforward to include the affects of variable density in the results of the previous sections. Variable density does not change the phase of the solution, but only the amplitude. Following Brekhovshikh [1980], if we assume that u denotes pressure, then the governing equation (1), is replaced by (op. cit., eq. (19.2))

$$\rho\nabla\cdot\ [(1/\rho)\nabla u] + \frac{\omega^2}{c^2} u = -\ \delta(x_1 - \xi_1)\ \delta(x_2)\ \delta(x_3 - \xi_3) \quad . \tag{64}$$

In this equation, $\rho = \rho(\underline{x})$ is the variable density.

We will again assume that u is of the form (2). However, in anticipation of a change in amplitude (but not in phase) we shall denote the amplitude by B:

$$u(\underline{x},\omega) \sim B(\underline{x})\ \exp\ \{i\omega\ \tau(\underline{x})\} \quad . \tag{65}$$

The phase τ is again a solution of the eikonal equation (3), but B is a solution of the transport equation

$$2\nabla B\cdot\nabla\tau + B\rho\nabla(1/\rho)\cdot\nabla\tau + B\nabla^2\tau = 0 \quad . \tag{66}$$

If we divide here by $\rho^{1/2}$, we find that

$$2\nabla(B\rho^{-1/2})\cdot\nabla\tau + B\rho^{-1/2}\,\nabla^2\tau = 0 \quad . \tag{67}$$

That is, $B\rho^{-1/2}$ satisfies the same transport equation as A does -- equation (4) -- and hence,

$$B(\underline{x}) = \text{const. } A(\underline{x})\sqrt{\rho(\underline{x})} \quad . \tag{68}$$

Here, const. means constant with respect to the running parameter along the rays. In fact, if one were given the same ray data for B as we were previously given for A, then the initial value of B would have to agree with the previous initial value of A. From this, it follows that

$$B(\underline{x}) = A(\underline{x})\sqrt{\rho(\underline{x})/\rho(\underline{x}_0)} \tag{69}$$

where $\underline{x}_0$ is the initial point on the ray.

The Green's function, developed in Section 2 is precisely a solution for which the initial data for A at $\underline{x} = \underline{\xi}$ -- equation (6) -- now becomes the initial data for B. Thus, from (12) we immediately determine that for 3D propagation

$$B = \frac{\sqrt{\rho(\underline{x}(\sigma,\alpha,\beta))}\ \sin\beta}{4\pi\sqrt{c(\underline{\xi})\ \rho(\underline{\xi})\ J(\sigma,\alpha,\beta)}} \tag{70}$$

with J again defined by (13).

In the specialization of Section 3 to the in-plane $2^1/_2$D Green's function, we find from (18) that

$$B = \frac{1}{4\pi} \sqrt{\frac{\rho(\underline{x}(\sigma,\beta))}{\sigma \; K\rho(\underline{\xi})}} \quad , \tag{71}$$

with K again given by (17) and $\underline{x}(\sigma,\beta)$ the solution of the ray equations in (14). Furthermore, the representation (19) is now replaced by

$$u \sim \frac{1}{4\pi} \sqrt{\frac{\rho(\underline{x}(\sigma,\beta))}{\sigma \; K \; \rho(\underline{\xi})}} \; \exp\{i\omega\tau\} \quad . \tag{72}$$

The amplitude B can be related to the amplitude B_2 of the two-dimensional asymptotic Green's function by (21) with A's replaced by B's. However, B_2 is related to A_2 by (69) with subscripts 2 introduced on both sides of the equations.

For the case in which ρ and c depend only on z or x_3, we use (32) for τ and (34) or (35) for K.

These results are to be substituted into (72) along with

$$\sqrt{\frac{\rho(\underline{x}(\sigma,\beta))}{\rho(\underline{\xi})}} = \sqrt{\frac{\rho(z)}{\rho(\xi_3)}} \quad . \tag{73}$$

The discussion of the in-plane Kirchhoff approximation in $2^1/_2$D,

presented in Section 4, must also be revised to allow for a jump in density as well as a jump in velocity. The reflection coefficient given in (46) was derived under the assumptions that u and $\partial u/\partial n$ are continuous across S. For the differential operator (64), we must replace those conditions by the conditions that u and $(1/\rho)\ \partial u/\partial n$ must be continuous across S. Proceeding as in the derivation of (46) given in [MMWP, Section 8.3], we deduce that the proper reflection coefficient is now given by

$$R = \frac{\rho_-^{-1}\ \partial\tau_I/\partial n - \rho_+^{-1}\ [\operatorname{sgn} \partial\tau_I/\partial n]\ \sqrt{c_+^{-2} - c_-^{-2} + [\partial\tau_I/\partial n]^2}}{\rho_-^{-1}\ \partial\tau_I/\partial n + \rho_+^{-1}[\operatorname{sgn} \partial\tau_I/\partial n]\ \sqrt{c_+^{-2} - c_-^{-2} + [\partial\tau_I/\partial n]^2}}. \tag{74}$$

In (49), we must replace $A(\underline{x};\xi)\ A(\underline{x};\eta)$ by $B(\underline{x};\xi)\ B(\underline{x};\eta)$ and use (74) for R. With these adjustments (58) is replaced by

$$u_S(\xi;\eta) \sim \frac{\sqrt{2\pi|\omega|}\ \exp\{3\pi i/4\ \operatorname{sgn}\omega\}}{(4\pi)^2\ \sqrt{\rho(\xi)\rho(\eta)}} \int_{C,\, x_3 = 0} \frac{R\rho_-(\underline{x})\ \exp\{i\omega[\tau(\underline{x};\xi) + \tau(\underline{x};\eta)]\}}{\sqrt{\sigma_\xi K_\xi \sigma_\eta K_\eta (\sigma_\xi^{-1} + \sigma_\eta^{-1})}} \tag{75}$$

$$\cdot\ \hat{n} \cdot [\nabla\tau(\underline{x};\xi) + \nabla\tau(\underline{x};\eta)\ ds \quad .$$

The specialization of this result to zero offset replaces equation (59):

$$u_S(\xi,;\xi) \sim \frac{\sqrt{|\omega|}\ \exp\{3\pi i/4 \operatorname{sgn}\omega\}}{8\pi^{3/2}\ \rho(\xi)} \int_{C,\, x_3 = 0} \frac{R\rho_-(\underline{x})\exp\{2i\omega\tau(\underline{x};\xi)\}}{K\sqrt{\sigma}}\ \hat{n}\cdot\nabla\tau(\underline{x};\xi)\ dS. \tag{76}$$

For the case of a depth dependent density function, we need only replace $\rho(\underline{\xi})$, $\rho(\underline{\eta})$ and $\rho(\underline{x})$ by $\rho(\xi_3)$ and $\rho(\eta_3)$ and $\rho(x_3)$, respectively. Furthermore, we use (32) and (34) for τ, σ, and K while still using (74) for R. These results can be used in (75) or (76).

This completes our discussion of the Variable Density case.

6. CONCLUSIONS

A few results about wave propagation in two-and-one-half dimensions have been derived. The first set have to do with the asymptotic radiation of acoustic waves from a point source. The second set are related to the Kirchhoff-approximate upward scattered field from a cylindrical surface at depth. In both cases, we have found that the in-plane propagation of a wave in three dimensions can be described totally in terms of in-plane calculations which are no more difficult than would be carried out to generate two dimensional models.

In our inversion research, we regularly develop our theory as if the observations are known over a planar array on the upper surface. However, we then specialize our results to the $2^1/_2$D case in which data is known only on a line. We have already begun using the results described here to generate synthetic data to test these $2^1/_2$D algorithms. This was the motivation for this project and the utility to our group has already been demonstrated.

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TABLE I

IN-PLANE GREEN'S FUNCTION

CONSTANT DENSITY

$c(x, z)$	$c(z)$	c_0
19, 17, 14	19, 32, 34	2, 24, 26

VARIABLE DENSITY

72, 17, 14	72, 32, 34, 73	

TABLE II

IN-PLANE KIRCHHOFF APPROXIMATE FIELD

CONSTANT DENSITY

	$c(x,z)$	$c(z)$	c_0
nonzero offset	58,46,17,14	58,46,34,32	60,61,62,23
zero offset	59,46,17,14	59,46,34,32	63,61,62,23

VARIABLE DENSITY

nonzero offset	75,74,17,14	75,74,34,32
zero offset	76,74,17,14	76,74,34,32

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ABSTRACT

The purpose of this paper is to collect certain wave propagation results in two-and-one-half dimensions -- defined as three dimensional propagation in a medium that has variations in two dimensions only. The results of interest are for sources and receivers in the plane determined by the two directions of parameter variation. The objective of this work is to reduce the analysis of the in-plane propagation to two dimensional analysis while retaining -- at least asympototically -- the proper three dimensional geometrical spreading. We do this for the free space Green's function and for the Kirchhoff approximate upward scattered field from a single reflector. In both cases, we carry out a derivation under the assumption of a background velocity with two dimensional -- c(x,z) -- variation; we specialize the results to a constant background velocity and a depth dependent background velocity. For the convenience of the user we have included a glossary and two tables of equation numbers to help in finding specific results.

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